REPORT No. 210

INERTIA FACTORS OF ELLIPSOIDS FOR USE IN AIRSHIP DESIGN

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This report is based on a study made by the writer as a member of the Special Committee on Design of Army Semirigid Airship RS–1 appointed by the National Advisory Committee for Aeronautics.

The increasing interest in airships has made the problem of the potential flow of a fluid about an ellipsoid of considerable practical importance. In 1833 Green,[1] in discussing the effect of the surrounding medium upon the period of a pendulum, derived three elliptic integrals, in terms of which practically all the characteristics of this type of motion can be expressed. The theory of this type of motion is very fully given by Lamb,[2] and applications to the theory of airships by many writers.[3] Tables of the inertia coefficients derived from these integrals are available for the most important special cases.[4] [5] These tables are adequate for most purposes, but occasionally it is desirable to know the values of these integrals in other cases where tabulated values are not available. For this reason it seemed worth while to assemble a collection of formulæ which would enable them to be computed directly from standard tables of elliptic integrals, circular and hyperbolic functions, and logarithms without the need of intermediate transformations. Some of the formulæ for special cases (elliptic cylinder, prolate spheroid, oblate spheroid, etc.) have been published before, but the general forms and some special cases have not been found in previous publications.

The additional inertia of the translational potential flow of a fluid about triaxial ellipsoid is proportional to the three coefficients

$$K_1=\frac{4\pi}{3}\,abc\,k_1,\ K_2=\frac{4\pi}{3}\,abc\,k_2, K_3=\frac{4\pi}{3}\,abc\,k_3$$

Here $\frac{4\pi}{3}\,abc$ is the volume of the ellipsoid and

$$k_1=\frac{\alpha_0}{2-\alpha_0},\ k_2=\frac{\beta_0}{2-\beta_0},\ k_3=\frac{\gamma_0}{2-\gamma_0}$$

The additional moment of inertia of the rotational potential flow is proportional to the three coefficients

$$K'_1=\frac{4\pi}{3}\,abc\,\frac{b^2+c^2}{5}\,k'_1, K'_2=\frac{4\pi}{3}\,abc\,\frac{c^2+a^2}{5}\,k'_2,\ K'_3=\frac{4\pi}{3}\,abc\,\frac{a^2+b^2}{5}\,k'_3$$

Here k'_1, k'_2, and k'_3 are given as factors of the corresponding moments of inertia of the ellipsoid itself and

$$k'_1=\left(\frac{b^2-c^2}{b^2+c^2}\right)^2\frac{\gamma_0-\beta_0}{2\,\frac{b^2-c^2}{b^2+c^2}-(\alpha_0-\beta_0)}$$

with symmetrical expressions for k'_2 and k'_3.

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[1] George Green: "Researches on the vibration of pendulums in fluid media." Trans. R. S. Ed. 1833.
[2] Horace Lamb: "Hydrodynamics" (4th ed. Camb. 1916), pp. 132–147.
[3] See, for example, Max M. Munk: "The aerodynamic forces on airship hulls." N. A. C. A., Report No. 184, 1924.
[4] Horace Lamb: "The inertia coefficients of an ellipsoid moving in fluid." G. B. A. C. A., R. & M. No. 623, 1918.
[5] H. Bateman: "The inertia coefficients of an airship in a frictionless fluid." N. A. C. A., Report No. 164, 1923.

In the above formulæ α_0, β_0, and γ_0 are the special values for $\lambda=0$ of Green's integrals

$$\alpha=abc\int_\lambda^\infty \frac{d\lambda}{(a^2+\lambda)\,\Delta},\ \beta=abc\int_\lambda^\infty \frac{d\lambda}{(b^2+\lambda)\,\Delta},\ \gamma=abc\int_\lambda^\infty \frac{d\lambda}{(c^2+\lambda)\,\Delta}$$

$$a\geqq b\geqq c \quad \Delta=\sqrt{(a^2+\lambda)\,(b^2+\lambda)\,(c^2+\lambda)}$$

To transform these integrals into the standard Legendre form substitute

$$\mathrm{sn}\,(u;k)=\mathrm{sn}u=\sqrt{\frac{a^2-c^2}{a^2+\lambda}},\ k^2=\frac{a^2-b^2}{a^2-c^2}<1,\ k'^2=\frac{b^2-c^2}{a^2-c^2}<1$$

This gives

$$a^2+\lambda=\frac{a^2-c^2}{\mathrm{sn}^2u},\ b^2+\lambda=(a^2-c^2)\,\frac{\mathrm{dn}^2u}{\mathrm{sn}^2u},\ c^2+\lambda=(a^2-c^2)\,\frac{\mathrm{cn}^2u}{\mathrm{sn}^2u}$$

and

$$\frac{d\lambda}{\Delta}=-\frac{2}{\sqrt{a^2-c^2}}\,du$$

Then

$$\alpha=\frac{2abc}{(a^2-c^2)^{3/2}}\int_0^u \mathrm{sn}^2u\,du=\frac{2abc}{(a^2-c^2)^{3/2}}\,\frac{1}{k^2}\,[u-E(u)]$$

$$\beta=\frac{2abc}{(a^2-c^2)^{3/2}}\int_0^u \frac{\mathrm{sn}^2u}{\mathrm{dn}^2u}\,du=\frac{2abc}{(a^2-c^2)^{3/2}}\,\frac{1}{k^2k'^2}\left[E(u)-k'^2u-k^2\,\frac{\mathrm{sn}u\ \mathrm{cn}u}{\mathrm{dn}\,u}\right]$$

$$\gamma=\frac{2abc}{(a^2-c^2)^{3/2}}\int_0^u \frac{\mathrm{sn}^2u}{\mathrm{dn}^2u}\,du=\frac{2abc}{(a^2-c^2)^{3/2}}\,\frac{1}{k'^2}\left[\frac{\mathrm{sn}u\ \mathrm{dn}u}{\mathrm{cn}u}-E(u)\right]$$

Here

$$\frac{\mathrm{sn}u\ \mathrm{dn}u}{\mathrm{cn}u}=\sqrt{\frac{(a^2-c^2)\,(b^2+\lambda)}{(a^2+\lambda)\,(c^2+\lambda)}},\ \frac{\mathrm{sn}u\ \mathrm{cn}u}{\mathrm{dn}u}=\sqrt{\frac{(a^2-c^2)\,(c^2+\lambda)}{(a^2+\lambda)\,(b^2+\lambda)}}$$

and

$$u=\mathrm{sn}^{-1}\sqrt{\frac{a^2-c^2}{a^2+\lambda}}=F\,(\varphi;\,k)\ \text{where}\ \varphi=\sin^{-1}\sqrt{\frac{a^2-c^2}{a^2+\lambda}}$$

The values of $u=F(\varphi;\,k)$ and $E(u)=E(\varphi;\,k)$ can be obtained directly from standard tables of elliptic integrals.

NOTE.—The notation of elliptic integrals is not standardized. Some authors write the elliptic integral of the second kind as a function of the amplitude φ. Some write the argument first and the modulus or modular angle second; some reverse the order, and some use one form at one time and another at another. Thus we may find the following forms:

$$u\equiv F\,(\varphi;\,k)\equiv F\,(k;\,\varphi)\equiv F\,(\varphi;\,\theta)\equiv F\,(\theta;\,\varphi)$$

$$E\,(u)\equiv E\,(u;\,k)\equiv E\,(u;\,\theta)\equiv E\,(\varphi;\,k)\equiv E\,(\varphi;\,\theta)\equiv E\,(k;\,\varphi)\equiv E\,(\theta;\,\varphi)$$

The more usual tables tabulate the functions according to the amplitude φ and the modular angle θ so that

$$u\equiv F\,(\varphi;\,\theta)\quad E\,(u)\equiv E\,(\varphi;\,\theta)$$

where

$$\varphi=\sin^{-1}\sqrt{\frac{a^2-c^2}{a^2+\lambda}},\quad \theta=\sin^{-1}\sqrt{\frac{a^2-b^2}{a^2-c^2}}$$

However, the latest, and for some purposes the most convenient, tables by R. L. Hippisley[6] tabulate $u=F\varphi=F\,(\varphi;\,\theta)$ and $E\,(u)=E\,(r)+eE$ according to r, where $r^0=90^0e=90^0\,\frac{u}{K}$.

[6] Smithsonian Mathematical Formulæ (1923), pp. 260-309.

When $\lambda=0$ the formulæ simplify to

$$\alpha_0=\frac{2abc}{(a^2-b^2)(a^2-c^2)^{1/2}}[u_0-E(u_0)]$$

$$\beta_0=\frac{2abc(a^2-c^2)^{1/2}}{(a^2-b^2)(b^2-c^2)}\left[E(u_0)-\frac{b^2-c^2}{a^2-c^2}u_0-\frac{(a^2-b^2)c}{ab(a^2-c^2)^{1/2}}\right]$$

$$\gamma_0=2\frac{1-\frac{ac}{b(a^2-c^2)^{1/2}}E(u_0)}{1-\left(\frac{c}{b}\right)^2}.$$

Here

$$\varphi_0=\sin^{-1}\frac{\sqrt{a^2-c^2}}{a}=\sin^{-1}e_1,\ u_0=F(\varphi_0;\ \theta)$$

$$\theta=\sin^{-1}\sqrt{\frac{a^2-b^2}{a^2-c^2}}=\sin^{-1}\frac{e_2}{e_1},\ E(u_0)=E(\varphi_0;\ \theta)$$

where e_1 and e_2 are the eccentricities of the central sections normal to the intermediate (b) and minimum (c) axes of the ellipsoid.

These formulæ are sufficient for the direct evaluation of k_1, k_2, k_3; k'_1, k'_2, and k'_3 in the general case. However, in special cases the elliptic integrals degenerate into algebraic, circular, hyperbolic, or other functions, or the coefficients take on indeterminate forms needing special treatment. The results for many of these special cases are more readily obtained by direct integration of the special differential forms, but for uniformity are discussed here as limiting forms of the general elliptic integrals.

1. Very long ellipsoid. Limiting case an elliptic cylinder. As a becomes large so that higher powers of both $\frac{c}{a}$ and $\frac{b}{a}$ become negligible $k\doteq 1$ and at the same time $\varphi_0\doteq\frac{\pi}{2}$.

$$u_0\doteq\log\frac{2a}{c}\text{ and }E(u_0)\doteq 1$$

In the limit since $x\log x\doteq 0$

$$\alpha_0=0,\ \beta_0=\frac{2}{1+\frac{b}{c}},\ \gamma_0=\frac{2}{1+\frac{c}{b}}$$

These are of course more directly obtained by treating the two dimensional flow around an elliptic cylinder.[7]

2. Elliptic disk. $c\doteq 0$. To quantities of the first order in c

$$\alpha_0=\frac{2c}{b(a^2-b^2)}[b^2u_0-b^2E(u_0)]$$

$$\beta_0=\frac{2c}{b(a^2-b^2)}[a^2E(u_0)-b^2u_0]$$

$$\gamma_0=2\left[1-\frac{c}{b}E(u_0)\right]$$

In the limit $c=0$, $\varphi_0=\frac{\pi}{2}$, so that $u_0=K$ and $E(u_0)=E$, the complete elliptic integrals, mod $\frac{\sqrt{a^2-b^2}}{a}=e$.

Then in the limit $\alpha_0=\beta_0=0$, $\gamma_0=2$, so that $k_1=k_2=0$, but $k_3=\infty$.
Thus $K_1=K_2=0$ and K_3 needs special evaluation:

$$K_3=\frac{4\pi}{3}abc\,k_3=\frac{4\pi}{3}abc\frac{\gamma_0}{2-\gamma_0}=\frac{4\pi}{3}abc\frac{1-\frac{c}{b}E(u_0)}{\frac{c}{b}E(u_0)}$$

[7] Horace Lamb, l. c., pp. 79-86.

In the limit $c=0$

$$K_3=\frac{4\pi}{3}\frac{ab^2}{E}, \bmod k=\frac{\sqrt{a^2-b^2}}{a}=e$$

when $a=b$ (circular plate) $k=e=0$, $E=\frac{\pi}{2}$, so that $K_3=\frac{8}{3}a^3$.

Again to quantities of the first order in c

$$k'_1=\frac{\gamma_0-\beta_0}{2-(\gamma_0-\beta_0)}$$

$$k'_2=\frac{\gamma_0-\alpha_0}{2-(\gamma_0-\alpha_0)}$$

$$k'_3=\left(\frac{a^2-b^2}{a^2+b^2}\right)^2\frac{\beta_0-\alpha_0}{\frac{a^2-b^2}{a^2+b^2}-(\beta_0-\alpha_0)}$$

In the limit $c=0$, $k'_3=0$, but k'_1 and k'_2 become infinite as $\frac{1}{c}$. To this order of approximation.

$$2-(\gamma_0-\beta_0)=2\frac{c}{b(a^2-b^2)}[(2a^2-b^2)\ E\ (u_0)-b^2u_0]$$

$$2-(\gamma_0-\alpha_0)=2\frac{c}{b(a^2-b^2)}[(a^2-2b^2)\ E\ (u_0)+b^2u_0]$$

so that when $c=0$

$$K'_1=\frac{4\pi}{15}\frac{ab^4\ (a^2-b^2)}{[(2a^2-b^2)\ E-b^2K]}$$

$$K'_2=\frac{4\pi}{15}\frac{a^3b^2(a^2-b^2)}{[(a^2-2b^2)\ E+b^2K]}$$

When $a\doteq b$ (circular disk), these become indeterminate, since $k\doteq 0$ and $E\doteq K\doteq\frac{\pi}{2}$. To quantities of the first order in (a^2-b^2), $(K-E)=\frac{\pi}{4}\frac{a^2-b^2}{a^2}$, so that $K'_1=K'_2=\frac{16}{45}a^5$.

3. Oblate spheroid. $a=b>c$, $k=0$, $k'=1$.

$$E\ (u)=u=\varphi=\sin^{-1}\sqrt{\frac{a^2-c^2}{a^2+\lambda}}=\sin^{-1}\frac{e}{\sqrt{1+\frac{\lambda}{a^2}}}$$

and $\lim\limits_{k\doteq 0}\frac{1}{k^2}[u-E\ (u)]=1/2\ (\varphi-\sin\varphi\cos\varphi)$

then

$$\alpha=\beta=\frac{2a^2c}{(a^2-c^2)^{3/2}}\ \frac{1}{2}(\varphi-\sin\varphi\cos\varphi)=\frac{2\sqrt{1-e^2}}{e^3}\frac{1}{2}\left(\varphi-\frac{e\sqrt{1-e^2}}{1+\frac{\lambda}{a^2}}\right)$$

$$\gamma=\frac{2a^2c}{(a^2-c^2)^{3/2}}(\tan\varphi-\varphi)=\frac{2\sqrt{1-e^2}}{e^3}\left(\frac{e}{\sqrt{1-e^2}}-\varphi\right)$$

When $\lambda=0$, $\varphi=\sin^{-1}e$, so that

$$\alpha_0=\beta_0=\frac{\sqrt{1-e^2}}{e^3}\left(\sin^{-1}e-e\sqrt{1-e^2}\right)$$

$$\gamma_0=2\frac{\sqrt{1-e^2}}{e^3}\left(\frac{e}{\sqrt{1-e^2}}-\sin^{-1}e\right)$$

In the limiting case $c=0$, $e=1$ (circular plate) these give as before

$$K_1=K_2=0, \quad K_3=\frac{8}{3}a^3$$

$$K'_1=K'_2=\frac{16}{45}a^5, K'_3=0$$

4. PROLATE SPHEROID. $a>b=c$, $k=1$, $k'=0$, $\varphi=\mathrm{gd}\,u$. Then

$$\alpha=\frac{2ac^2}{(a^2-c^2)^{3/2}}(u-\tanh u)$$

$$\beta=\gamma=\frac{2ac^2}{(a^2-c^2)^{3/2}}1/2\ (\sinh u \cosh u-u)$$

where

$$\tanh u=\sin\varphi=\sqrt{\frac{a^2-c^2}{a^2+\lambda}}=\frac{e}{\sqrt{1+\frac{\lambda}{a^2}}}, \quad \frac{ac^2}{(a^2-c^2)^{3/2}}=\frac{1-e^2}{e^3}$$

$$\sinh u\cosh u=\frac{\sqrt{(a^2-c^2)\ (a+\lambda)}}{c^2+\lambda}=\frac{e^2\sqrt{1+\frac{\lambda}{a^2}}}{1-e^2+\frac{\lambda}{a^2}}$$

and

$$u=\log\sqrt{\frac{1+\sin\varphi}{1-\sin\varphi}}=\log\tan\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)$$

when $\lambda=0$, these reduce to

$$\alpha=\frac{(1-e^2)}{e^3}\left[\log\frac{1+e}{1-e}-2e\right]$$

$$\beta=\gamma=\frac{(1-e^2)}{e^3}\left[\frac{e}{1-e^2}-1/2\log\frac{1+e}{1-e}\right]$$

The special cases 3 and 4 are of course more readily obtained by direct integration.